Combinatorics on Brauer-type semigroups

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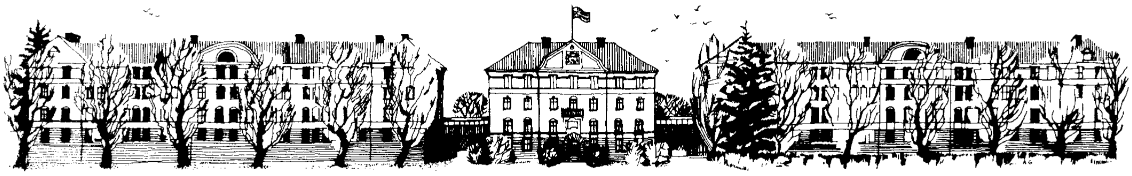
U.U.D.M. Project Report 2006:3

Examensarbete i matematik, 20 poäng

Handledare och examinator: Volodymyr Mazorchuk

Maj 2006



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Abstract

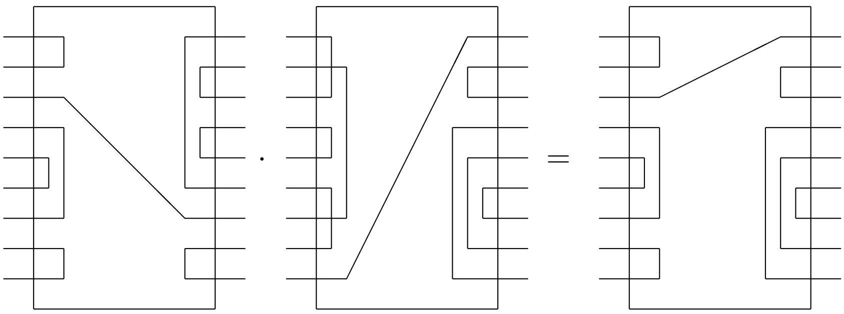
The Brauer semigroup $\mathfrak{B}_n$ of partitions of a $2n$-element set into two-element subsets, can be generalized in various ways. For example we may consider arbitrary partitions to obtain the semigroup $\mathfrak{C}_n$. The visualization of elements of these semigroups as chips leads us to the Temperley-Lieb semigroup $\mathfrak{TL}_n$ and its analogue $\mathfrak{TLC}_n$. We study combinatorial properties of these four semigroups. For example, we describe Green's relations for these semigroups and we count the size of the various equivalence classes. We also study idempotents and the corresponding maximal subgroups in these semigroups, and give some formulas for calculating the number of idempotents. Finally we study regularity and inverse elements of our four semigroups.

1 Introduction

Let $n \in \mathbb{N}$ and let $N = \{1, 2, \dots, n\}$ and $N^\circ = \{1^\circ, 2^\circ, \dots, n^\circ\}$. Then the *Brauer semigroup* $\mathfrak{B}_n$ consists of all possible partitions of $N \cup N^\circ$ into two-element subsets. We will often view elements of $\mathfrak{B}_n$ as microchips with n pins to the left representing the elements of N, which we will call initial elements, and n pins to the right representing the elements of N°, the terminal elements. Furthermore, for each set in the partition we draw a line between the two corresponding pins in the picture.

Let $\alpha, \beta \in \mathfrak{B}_n$. Then a sequence $x_1, x_2, \dots, x_k \in N$ is called $\alpha - \beta$ l-connected if $\{x_{2l-1}, x_{2l}\} \in \beta$ and $\{x^\circ_{2l}, x^\circ_{2l+1}\} \in \alpha$ for all $0 < l < \frac{k}{2}$. The sequence is called $\alpha - \beta$ r-connected if $\{x^\circ_{2l-1}, x^\circ_{2l}\} \in \alpha$ and $\{x_{2l}, x_{2l+1}\} \in \beta$ for all $0 < l < \frac{k}{2}$. The multiplication of two elements $\alpha, \beta \in \mathfrak{B}_n$ is now defined in the following way: For two elements $i, j \in N$ we have $\{i, j^\circ\} \in \alpha\beta$ if and only if there is an $\alpha - \beta$ l-connected sequence $x_1, x_2, \dots, x_{2k+1} \in N$ of odd length such that $\{i, x^\circ_1\} \in \alpha$ and $\{x_{2k+1}, j^\circ\} \in \beta$. Furthermore, $\{i, j\} \in \alpha\beta$ if and only if either $\{i, j\} \in \alpha$ or there is an $\alpha - \beta$ l-connected sequence $x_1, x_2, \dots, x_{2k} \in N$ of even length such that $\{i, x^\circ_1\}, \{j, x^\circ_{2k}\} \in \alpha$. Finally,

Figure 1: Multiplication of chips in $\mathfrak{B}_n$



$\{i^\circ, j^\circ\} \in \alpha\beta$ if and only if either $\{i^\circ, j^\circ\} \in \beta$ or there is an $\alpha - \beta$ r-connected sequence $x_1, x_2, \ldots, x_{2k} \in N$ of even length such that $\{x_1, i^\circ\}, \{x_{2k}, j^\circ\} \in \beta$.

The multiplication of two elements of $\mathfrak{B}_n$ can informally be viewed as connection of pins, in the following way: If $\alpha, \beta \in \mathfrak{B}_n$ then $\alpha\beta$ is the chip we get by connecting the right pins of α and the left pins of β and omitting the so-called dead circles which may appear, see figure 1 for an example. From this we also see that the multiplication is associative.

Note that the lines in the chip picture may cross each other. This leads us to the definition of the *Temperley-Lieb semigroup* $\mathfrak{TL}_n$, which consists of those elements of $\mathfrak{B}_n$, whose chip picture can be drawn without any lines crossing each other. The product is the same as in $\mathfrak{B}_n$ and it is easy to see that $\mathfrak{TL}_n$ is closed under this multiplication. In figure 1, the leftmost and rightmost elements are in $\mathfrak{TL}_n$, while the element in the middle is not.

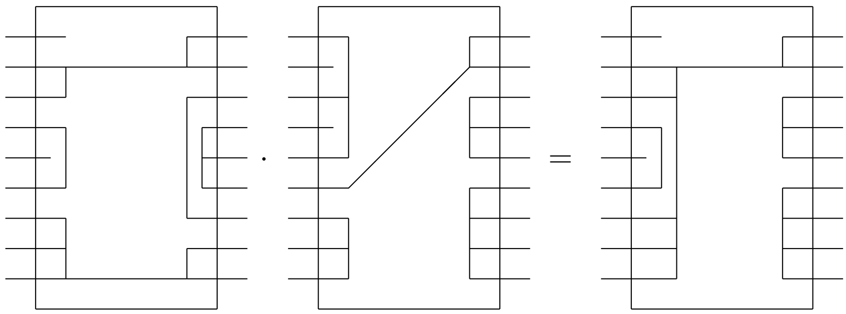
In the construction of $\mathfrak{B}_n$ we only allow two-element subsets in the partition. An obvious generalization of this concept is to allow partitions consisting of arbitrarily big sets. Let N and N° be as above. Then we define $\mathfrak{C}_n$ to be the set of all partitions of $N \cup N^\circ$. Elements of $\mathfrak{C}_n$ can be drawn as chips in the same way as for $\mathfrak{B}_n$, and the product is analogous to the multiplication in that semigroup, see figure 2 for an example.

As in the case of $\mathfrak{B}_n$ we may consider a subset of chips which can be drawn without any lines crossing each other. Multiplying two such elements yields again an element of the same type. Since multiplication in $\mathfrak{C}_n$ is associative we get a semigroup which we denote by $\mathfrak{TLC}_n$, in analogy to the Temperley-Lieb semigroup. All chips in figure 2 are elements in $\mathfrak{TLC}_n$.

On all the above defined semigroups we can define an involution which we will call σ, which interchanges initial and terminal elements, i.e. interchanges x and x° for all $x \in N$.

Let $a \in \mathfrak{B}_n$ and let N and N° be the initial respectively terminal elements

Figure 2: Multiplication of chips in $\mathfrak{C}_n$



of a. Then the set of initial elements connected with terminal elements is called the initial image of a and is denoted by $\mathrm{Im}(a)$. The set of terminal elements connected with initial elements is called the terminal image and is denoted by $\mathrm{Im}^\circ(a)$. Furthermore the set of initial elements connected with initial elements is called the initial kernel, denoted by $\mathrm{Ker}(a)$, and the set of terminal elements connected with terminal elements is called the terminal kernel, denoted by $\mathrm{Ker}^\circ(a)$.

2 Cardinalities of $\mathfrak{B}_n$, $\mathfrak{TL}_n$, $\mathfrak{C}_n$ and $\mathfrak{TLC}_n$

In this section we will begin our studies of combinatorial properties of the semigroups defined above by investigating cardinalities.

Proposition 2.1. $|\mathfrak{B}_n| = (2n-1)!!$.

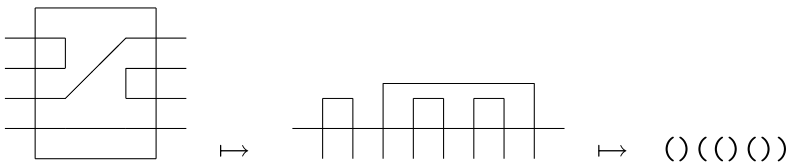
Proof. We choose two-element subsets of $\{1, 2, \ldots, n, 1', 2', \ldots, n'\}$. For each choice we get 2 elements less to choose from. Hence we can do this in $\binom{2n}{2}\binom{2n-2}{2}\cdot\ldots\cdot\binom{2}{2}$ different ways. The number of partitions does not depend on order of choice of the n subsets, so we have to divide by $n!$. Furthermore we have $\binom{2k}{2} = 2k^2 - k$ and hence

$$|\mathfrak{B}_n| = \frac{1}{n!}\prod_{k=1}^{n}\binom{2k}{2} = \prod_{k=1}^{n}\frac{1}{k}(2k^2-k) = \prod_{k=1}^{n} 2k-1 = (2n-1)!!.$$

□

Proposition 2.2. $|\mathfrak{C}_n| = \sum_{k=1}^{2n} S_{2n,k}$*, where* $S_{n,k}$ *are the Stirling numbers of second kind.*

Figure 3: Correspondence between $\mathfrak{TL}_n$ and $\mathcal{W}_n$



Proof. The number of ways to partition a $2n$-element set into k blocks equals $S_{2n,k}$. Hence we only need to sum over the number of blocks and the result follows. □

Before studying the cardinalities of $\mathfrak{TL}_n$ and $\mathfrak{TLC}_n$ we need to consider well-formed expressions of parentheses. An expression consisting of n left and n right parentheses is called *well-formed* if for any $0 \leq k \leq 2n$ the substring consisting of the first k parentheses in the expression contains at least as many left parentheses as right parentheses. We will use $\mathcal{W}_n$ to denote the set of all such expressions.

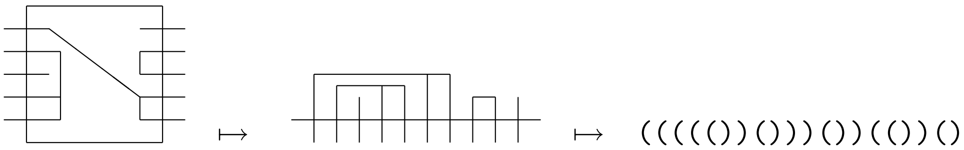
Lemma 2.3. *The set $\mathcal{W}_n$ of well-formed expressions of n left and n right parentheses has cardinality C_n, the n-th Catalan number defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

Proof. Let E be such an expression and let us consider a square with side n. Reading E from left to right we perform a walk from the left lower corner to the right upper corner in the following way: If we read a left parenthesis we go one step to the right. If we read a right parenthesis we go one step upwards. Since E contains n left and n right parentheses we will arrive at the upper right corner, and since we walk only to the right and upwards all such paths are the shortest along perpendicular lines. Furthermore, since E is well-formed, we do not walk above the diagonal. Conversely, if we have such a shortest path we may construct an expression in $\mathcal{W}_n$ by writing left respectively right parentheses when the path goes to the right respectively upwards. Hence we have a bijection between expressions in $\mathcal{W}_n$ and paths below the diagonal. It is well-known that the number of such paths is the n-th Catalan number C_n. □

Proposition 2.4. $|\mathfrak{TL}_n| = C_n$*, where C_n is the n-th Catalan number.*

Proof. Let us first modify the chip picture of an element in $\mathfrak{TL}_n$ by putting the pins of the chip on a straight line, as in figure 3. It is obvious that this does not change the property of non-crossing lines. This picture can now be

Figure 4: Correspondence between $\mathfrak{TLC}_n$ and $\mathcal{W}_{2n}$



replaced bijectively by a well-formed expression of left and right parentheses in the following way: Read the picture from the left. For every point where a line begins and goes to the right, write a left parenthesis. For every point where a line comes from the left and ends, write a right parenthesis. This algorithm can easily be reversed so we have a bijection. By the lemma, the number of such expressions is C_n. □

Proposition 2.5. *$|\mathfrak{TLC}_n| = C_{2n}$, the $2n$-th Catalan number.*

Proof. First redraw the chip picture as in the previous proof. We construct a bijection between $\mathfrak{TLC}_n$ and the set $\mathcal{W}_{2n}$ of all well-formed expressions of $2n$ left and $2n$ right parentheses in the following way: Replace the left endpoint of a bracket with `((`, the right endpoint with `))`, a middle pin with `)(` and a pin not connected to any bracket with `()`. In this way we get a well-formed expression of parentheses.

Conversely, let $e \in \mathcal{W}_{2n}$ be arbitrary. Divide the expression e into n pairs of parentheses. The four pairs which can arise are then `((`, `()`, `)(` and `))`. By applying the rules above backwards we construct a chip. Since $e \in \mathcal{W}_{2n}$ we have the following:

(i) The pair `()` may occur everywhere in the expression, since the initial left parenthesis immediately is closed by the right parenthesis. This shows that a pin not connected to any other pin may occur everywhere in the chip.

(ii) The number of `((`-pairs is the same as the number of `))`-pairs since the two other types of pairs has the same number of right parentheses as left parenthesis. Since these types of pairs begins and finishes a bracket, this shows that an opened bracket always will be closed.

(iii) When a `))`-pair occurs in the expression, it must have occurred at least as many `((`-pairs as `))`-pairs before it, since otherwise we have too many right parenthesis in an initial substring of the expression. This shows that it is not possible to close a non-opened bracket. Any two brackets do not cross each other since the pin corresponding to the first `((`-pair is connected to the pin corresponding to the last `))`-pair etc.

(iv) When a `)(`-pair occurs it must have occurred at least as many `((`-pairs as `))`-pairs, since otherwise we have too many right parenthesis in an initial substring of the expression. This shows that there must exist a bracket to which we can connect a middle pin. A middle pin is always connected to the undermost bracket.

Hence every such expression yields an element of $\mathfrak{TLC}_n$, and hence the statement follows from the Lemma. □

3 Green's relations

An important part of the study of a certain semigroup is to determine Green's relations on it. In this section we will study Green's relations on the four semigroups $\mathfrak{B}_n$, $\mathfrak{TL}_n$, $\mathfrak{C}_n$ and $\mathfrak{TLC}_n$. But first, before specifying the semigroups, let us study some general theory of semigroups and Green's relations on them. In this first section we follow [1] and [3].

A binary relation, here often called just a relation, on a set X is a subset $\rho \subset X \times X$, and if $(a,b) \in R$ we can write aRb. If ρ and σ are two relations on X we define their composition $\rho \circ \sigma$ to be the relation τ on X such that $(a,b) \in \tau$ if there is an element $x \in X$ such that $(a,x) \in \rho$ and $(x,b) \in \sigma$. This composition is associative and hence the set of binary relations forms a semigroup together with $\circ$. Furthermore, define $\epsilon = \{(x,x), x \in X\}$ and for a relation ρ define $\rho^{-1} = \{(x,y) \in X \times X \colon (y,x) \in \rho\}$. Since relations are subsets we also can form the intersection of two relations. If $\tau = \rho \cap \sigma$ then $(a,b) \in \tau$ if $(a,b) \in \rho$ and $(a,b) \in \sigma$.

In the above terminology the relation ρ on S is an equivalence relation if $\epsilon \subset \rho$ (reflexivity), $\rho \subset \rho^{-1}$ (symmetry) and $\rho \circ \rho \subset \rho$ (transitivity). An equivalence relation is called a left congruence if $a\rho b$ implies $ca\rho cb$, a right congruence if $a\rho b$ implies $ac\rho bc$, and just a congruence if it is both a left and a right congruence.

Let S is a semigroup. For two subsets $A, B \in S$ we define the product $AB = \{ab \colon a \in A, b \in B\}$. Then a left ideal of S is a non-empty subset $A \subset S$ such that $SA \subset A$. In the same way, a right ideal of S is a non-empty subset $A \subset S$ such that $AS \subset A$. A two-sided ideal is a subset $A \subset S$ which is both a left and a right ideal.

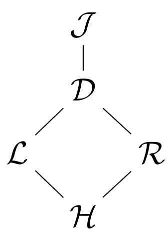
Let A be a nonempty subset of S. The left ideal generated by A is the intersection of all left ideals containing A. In the case of a semigroup S this equals $A \cup SA$. Here it may be useful to consider the notion of adjoining an identity element to the semigroup S if necessary. Let $\mathbf{1} \notin S$ and define $\mathbf{1} \cdot \mathbf{1} = \mathbf{1}$ and $\mathbf{1} \cdot a = a \cdot \mathbf{1} = a$ for all $a \in S$. Then define

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element,} \\ S \cup \mathbf{1} & \text{otherwise.} \end{cases}$$

Using this new notation, the left ideal generated by the set A is S^1A, the right ideal generated by A is AS^1, and the two-sided ideal generated by A is S^1AS^1. If $A = \{a\}$ we get the principal left ideal S^1a, the principal right ideal aS^1 and the principal two-sided ideal S^1aS^1.

Now we are ready to introduce the five Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ on the semigroup S. Two elements $a, b \in S$ are $\mathcal{L}$-related, $a\mathcal{L}b$, if they generate the same left principal ideal of S, i.e. if $S^1a = S^1b$. In a dual way $a, b \in S$ are $\mathcal{R}$-related, $a\mathcal{R}b$, if they generate the same right principal ideal of S, i.e. if $aS^1 = bS^1$. From these definitions we now define $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$, i.e. $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$, and $a\mathcal{D}b$ if and only if there exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$. Finally, $a, b \in S$ are $\mathcal{J}$-related, $a\mathcal{J}b$, if they generate the same two-sided principal ideal of S, i.e. if $S^1aS^1 = S^1bS^1$.

Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ we immediately have the inclusions $\mathcal{H} \subset \mathcal{L}$ and $\mathcal{H} \subset \mathcal{R}$. Furthermore, $\mathcal{L} \subset \mathcal{D}$ by using reflexivity of $\mathcal{R}$. Analogously $\mathcal{R} \subset \mathcal{D}$. Since $\mathcal{L} \subset \mathcal{J}$ and $\mathcal{R} \subset \mathcal{J}$ by definition, we also have $\mathcal{D} \subset \mathcal{J}$. Hence we have the following inclusion diagram for Green's relations:



In a finite semigroups the uppermost inclusion turns out to be an equality:

Proposition 3.1. *Let S be a finite semigroup. Then the relations $\mathcal{D}$ and $\mathcal{J}$ coincides on S.*

Proof. Since $\mathcal{D} \subset \mathcal{J}$ we only need to prove $\mathcal{J} \subset \mathcal{D}$. So let $(a, b) \in \mathcal{J}$. Then, by definition, there exists $x, y, w, z \in S^1$ such that $xay = b$ and $wbz = a$. Substituting the first equality into the second we get $wxayz = a$. Thus we have $(wx)^n a (yz)^n = a$ for all $n \in \mathbb{N}$. Now, since S is finite the elements in the sequence $(s^n)_{n \in \mathbb{N}}$ cannot be pairwise different for $s \in S$. Thus there must exist $i, j \in \mathbb{N}$ such that $s^i = s^{i+j}$ and hence s^j is an idempotent. In our situation there exist $k, l \in \mathbb{N}$ such that $e = (wx)^k$ and $f = (yz)^l$ are idempotents. Now $a = (wx)^{kl} a (yz)^{kl} = eaf$, so it follows that $(wx)^k a =$

$ea = a$ and $a(yz)^l = af = a$. Thus $xa\mathcal{L}a$ and $ay\mathcal{R}a$. Furthermore, since $\mathcal{L}$ is a right congruence we have $xay\mathcal{L}ay$ and then $b\mathcal{L}ay$. Hence $(a, b) \in \mathcal{D}$, so $\mathcal{J} \subset \mathcal{D}$ and the statement is proved. □

Let $a \in S$. Then let $\mathcal{L}_a$ be the equivalence class of $\mathcal{L}$ containing a, i.e. $\mathcal{L}_a = \{s \in S\colon s\mathcal{L}a\}$. Similarly we let $\mathcal{R}_a$, $\mathcal{H}_a$, $\mathcal{D}_a$ and $\mathcal{J}_a$ be the $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class and $\mathcal{J}$-class of a respectively. We now have the following fundamental statement, due to Green, in particular stating that two $\mathcal{L}$-classes in the same $\mathcal{D}$-class have the same size.

Lemma 3.2 (Green's lemma). *Let S be a semigroup and $a, b \in S$ two $\mathcal{L}$-equivalent elements. Let $s, t \in S^1$ be elements such that $sa = b$ and $tb = a$ and define the mappings $\sigma\colon x \mapsto sx$ and $\tau\colon x \mapsto tx$. Then σ and τ are mutually inverse $\mathcal{L}$-class preserving bijections between $\mathcal{R}_a$ and $\mathcal{R}_b$.*

Proof. Let $x \in \mathcal{R}_a$. Since $\mathcal{R}$ is a left congruence, $x\mathcal{R}a$ implies $\sigma(x) = sx\mathcal{R}sa = b$, and thus $\sigma(x) \in \mathcal{R}_b$. Hence σ maps $\mathcal{R}_a$ to $\mathcal{R}_b$. Furthermore τ maps $\mathcal{R}_b$ to $\mathcal{R}_a$ by a similar argument.

For $x \in \mathcal{R}_a$ there exists $r \in S^1$ such that $x = ar$ and we have

$$\tau(\sigma(x)) = \tau(sx) = tsx = tsar = tbr = ar = x,$$

showing that $\tau \circ \sigma$ is the identity map on $\mathcal{R}_a$. For $y \in \mathcal{R}_b$ there exists $q \in S^1$ such that $y = bq$ and we have

$$\sigma(\tau(y)) = \sigma(ty) = sty = stbq = saq = bq = y,$$

showing that $\sigma \circ \tau$ is the identity map on $\mathcal{R}_b$. Hence σ and τ are inverse bijections.

Let $x, y \in \mathcal{R}_a$ and $x\mathcal{L}y$. Since $x = \tau(\sigma(x)) = t\sigma(x)$ we have $x\mathcal{L}\sigma(x)$, and similarly $y\mathcal{L}\sigma(y)$. Thus $\sigma(x)\mathcal{L}x$, $x\mathcal{L}y$ and $y\mathcal{L}\sigma(y)$ and by transitivity $\sigma(x)\mathcal{L}\sigma(y)$ and hence σ preserves $\mathcal{L}$-classes. Again, a similar argument shows that τ preserves $\mathcal{L}$-classes. □

There is a dual statement of Green's lemma, given by interchanging left and right:

Lemma 3.3 (Green's lemma, left-right dual). *Let S be a semigroup and $a, b \in S$ two $\mathcal{R}$-equivalent elements. Let $s, t \in S^1$ be elements such that $as = b$ and $bt = a$ and define the mappings $\sigma\colon x \mapsto xs$ and $\tau\colon x \mapsto xt$. Then σ and τ are mutually inverse $\mathcal{R}$-class preserving bijections between $\mathcal{L}_a$ and $\mathcal{L}_b$.*

Proof. Green's lemma (Lemma 3.2) is valid for all semigroups $S = (S, \cdot)$. Define the opposite semigroup $S^{op} = (S, *)$ of S by letting $a * b = b \cdot a$ for $a, b \in S$. One can easily see that the left Green relation $\mathcal{L}$ on S becomes the right Green relation $\mathcal{R}$ on S^{op}, and vice versa. Hence the dual statement is valid on S^{op} if and only if Green's lemma is valid. But every semigroup is obviously an opposite semigroup of some semigroup, and hence the dual statement follows from the original one. □

Putting these two statements together we get the following proposition, telling us that all $\mathcal{H}$-classes in a $\mathcal{D}$-class have the same cardinality:

Proposition 3.4. *Let S be a semigroup and $a, b \in S$ two $\mathcal{D}$-equivalent elements. Let $c \in S$ be such that $a\mathcal{L}c$ and $c\mathcal{R}b$, so there exists $s_l, s_r, t_l, t_r \in S^1$ such that $s_l a = c$, $t_l c = a$, $cs_r = b$ and $bt_r = c$ and define the mappings $\sigma \colon x \mapsto s_l x s_r$ and $\tau \colon x \mapsto t_l x t_r$. Then σ and τ are mutually inverse bijections between $\mathcal{H}_a$ and $\mathcal{H}_b$.*

Proof. By Lemma 3.2, the mappings $\sigma_l \colon x \mapsto s_l x$ and $\tau_l \colon x \mapsto t_l x$ are mutually inverse $\mathcal{L}$-class preserving bijections between $\mathcal{R}_a$ and $\mathcal{R}_c$, and by Lemma 3.3, the mappings $\sigma_r \colon x \mapsto x s_r$ and $\tau_r \colon x \mapsto x t_r$ are mutually inverse $\mathcal{R}$-class preserving bijections between $\mathcal{L}_c$ and $\mathcal{L}_b$. Now observe that if $x \in \mathcal{H}_a$ then $x \in \mathcal{L}_a$ and, since σ_l is $\mathcal{L}$-class preserving, $\sigma_l(x) \in \mathcal{L}_{\sigma(a)} = \mathcal{L}_c$, so $\sigma_l(x) \in \mathcal{H}_c$. By similar arguments we have analogous conclusions for the other mappings. Therefore we may define $\bar{\sigma}_l \colon \mathcal{H}_a \to \mathcal{H}_c$ and $\bar{\tau}_l \colon \mathcal{H}_c \to \mathcal{H}_a$ to be the restrictions of σ_l and τ_l to $\mathcal{H}_a$ and $\mathcal{H}_c$ respectively, and $\bar{\sigma}_r \colon \mathcal{H}_c \to \mathcal{H}_b$ and $\bar{\tau}_r \colon \mathcal{H}_b \to \mathcal{H}_c$ to be the restrictions of σ_r and τ_r to $\mathcal{H}_c$ and $\mathcal{H}_b$ respectively. Then σ_l, τ_l and σ_r, τ_r are mutually inverse maps on $\mathcal{H}_a$, $\mathcal{H}_c$ and $\mathcal{H}_c$, $\mathcal{H}_b$ respectively. Hence their compositions $\sigma_r \circ \sigma_l$ and $\tau_l \circ \tau_r$ are mutually inverse bijections of $\mathcal{H}_a$ and $\mathcal{H}_b$ upon each other. But $\sigma_r \circ \sigma_l(x) = s_l x s_r$ and $\tau_l \circ \tau_r(x) = t_l y t_r$ so $\sigma_r \circ \sigma_l = \sigma$ and $\tau_l \circ \tau_r = \tau$ and hence the statement is proved. □

After this general part of theory, let us now study Green's relations on our four specific semigroups. In this section we generalize the statements on Green's relations stated for $\mathfrak{B}_n$ in [5] to statements on Green's relations on $\mathfrak{TL}_n$, $\mathfrak{C}_n$ and $\mathfrak{TLC}_n$. We will also investigate some combinatorial properties of the equivalence classes of Green's relations on these semigroups.

First let us introduce the notation of neighbours used in [5]. Two elements $a, b \in \mathfrak{B}_n$ or $a, b \in \mathfrak{TL}_n$ are called left neighbours provided $\{x, y\} \in a$ if and only if $\{x, y\} \in b$. a and b are called right neighbours provided $\{x^\circ, y^\circ\} \in a$ if and only if $\{x^\circ, y^\circ\} \in b$. If a and b are both left and right neighbours they are called just neighbours. For analogy reasons we also use $l(a)$ to denote the number of sets in the partition consisting of both initial and terminal elements. This can be viewed as the number of lines between the left and right hand side of the chip, and for $a \in \mathfrak{B}_n$ or $a \in \mathfrak{TL}_n$ we have $l(a) = |\operatorname{Im}(a)|$. Henceforth we will call a set consisting of both initial and terminal elements a line, a set consisting of only initial elements we call an initial bracket and a set consisting of only terminal elements we call a terminal bracket.

Theorem 3.5. *Let $a, b \in \mathfrak{B}_n$ or $a, b \in \mathfrak{TL}_n$. Then*

(i) $a\mathcal{L}b$ if and only if a and b are right neighbours,

(ii) $a\mathcal{R}b$ if and only if a and b are left neighbours,

(iii) $a\mathcal{H}b$ if and only if a and b are neighbours,

(iv) $a\mathcal{D}b$ if and only if $l(a) = l(b)$.

Proof. Recall that, for a semigroup S, $a\mathcal{L}b$ if and only if $S^1a = S^1b$ which is equivalent to existence of elements $c, d \in S$ such that $ca = b$ and $db = a$. The necessity follows immediately since for example in (i) the right hand side of a generating element is preserved in all elements of the generated ideal. Furthermore sufficiency of (iii) follows from sufficiency of (i) and (ii), and sufficiency of (i) follows from that of (ii) by interchanging the initial and terminal elements $x \leftrightarrow x^\circ$. In case of (iv), $a\mathcal{D}b$ implies existence of c such that $a\mathcal{L}c$ and $c\mathcal{L}b$. By necessity of (i) and (ii), a and c are right neighbours, and c and b are left neighbours, which implies $l(a) = l(c) = l(b)$.

For sufficiency in (i) it is sufficient to prove that for any left neighbours $a, b \in \mathfrak{B}_n$ we have $a \in b\mathfrak{B}_n$, and for any left neighbours $a, b \in \mathfrak{TL}_n$ we have $a \in b\mathfrak{TL}_n$. We prove this by construction of an element $c \in \mathfrak{B}_n$ respectively $c \in \mathfrak{TL}_n$ such that $a = bc$. The element c is constructed by "copying" the structure of the terminal elements of b to the initial elements of c and the structure of terminal elements of a to the terminal elements of c in the following sense: For any set $\{x^\circ, y^\circ\} \in b$ we have a set $\{x, y\} \in c$, and for any set $\{x^\circ, y^\circ\} \in a$ we have a set $\{x^\circ, y^\circ\} \in c$. Furthermore, if $\{x, y^\circ\} \in b$ and $\{x, z^\circ\} \in a$, then we have $\{y, z^\circ\} \in a$, and if $a, b \in \mathfrak{TL}_n$ then $c \in \mathfrak{TL}_n$. This element c satisfies the above equality, and hence (i) is proved.

The sufficiency proof for (ii) is done analogously as for (i). Sufficiency of (iii) then follows from definitions and that of (i) and (ii). In (iv) we need to show existence of an element c such that $a\mathcal{L}c$ and $c\mathcal{L}b$. But since $l(a) = l(b)$ it is always possible to construct an element which is a right neighbour of a and a left neighbour of c. □

This theorem gives the information needed to calculate cardinalities of the equivalence classes, which we state in the following corollaries:

Corollary 3.6. *Let* $\pi \in \mathfrak{B}_n$*. Then*

(i) $|\mathcal{L}_\pi| = |\mathcal{R}_\pi| = l(\pi)!\binom{n}{l(\pi)}(n - l(\pi) - 1)!!$,

(ii) $|\mathcal{H}_\pi| = l(\pi)!$,

(iii) $|\mathcal{D}_\pi| = l(\pi)!\left(\binom{n}{l(\pi)}(n - l(\pi) - 1)!!\right)^2$.

Proof. As we have seen above, there exists an involution σ which flips the chip. It swaps the relations $\mathcal{L}$ and $\mathcal{R}$, and hence the statement for $\mathcal{R}$ follows from the statement for $\mathcal{L}$. We know that $\tau\mathcal{L}\pi$ iff τ and π are right neighbours, i.e. $\mathrm{Ker}^\circ(\tau) = \mathrm{Ker}^\circ(\pi)$. To construct τ we have to choose $\mathrm{Ker}(\tau) \subset \mathrm{In}(\tau)$ and thus also choose $\mathrm{Im}(\tau) \subset \mathrm{In}(\tau)$ which can be done in $\binom{n}{l(\pi)}$ different ways. $\mathrm{Ker}(\tau)$ can be partitioned into two-element subsets in $|\mathfrak{B}_{(n-l(\pi))/2}| = (n - l(\pi) - 1)!!$ different ways. To complete the chip we have to connect pins in $\mathrm{Im}(\tau)$ with pins in $\mathrm{Im}^\circ(\tau)$ which can be done in $(n - l(\pi))!$ different ways. Hence the result follows.

For the proof of the second statement we note that if $\tau\mathcal{H}\pi$, then $\mathrm{Ker}(\tau) = \mathrm{Ker}(\pi)$ and $\mathrm{Ker}^\circ(\tau) = \mathrm{Ker}^\circ(\pi)$. To complete the chip we need to connect pins from $\mathrm{Im}(\tau)$ and from $\mathrm{Im}^\circ(\tau)$, which can be done in $l(\pi)!$ different ways.

To construct a τ for which $\tau\mathcal{D}\pi$ we choose and partition $\mathrm{Ker}(\tau) \subset \mathrm{In}(\tau)$, and choose and partition $\mathrm{Ker}^\circ(\tau) \subset \mathrm{Ter}(\tau)$, both choices can be done in $\binom{n}{l(\pi)}(n - l(\pi) - 1)!!$ different ways as above. Finally we choose connections between $\mathrm{Im}(\tau)$ and $\mathrm{Im}^\circ(\tau)$ as above, and hence we get the desired result. □

Corollary 3.7. *Let* $\pi \in \mathfrak{TL}_n$ *with* $l(\pi) = n$*. Then*

$$|\mathcal{L}_\pi| = |\mathcal{R}_\pi| = |\mathcal{H}_\pi| = |\mathcal{D}_\pi| = 1.$$

Proof. Immediate since there is only one such π with $l(\pi) = n$, namely the identity element. □

Corollary 3.8. *Let* $\pi \in \mathfrak{TL}_n$ *with* $0 \le l(\pi) < n$ *and let* $k = \frac{n-l(\pi)}{2}$*. Then*

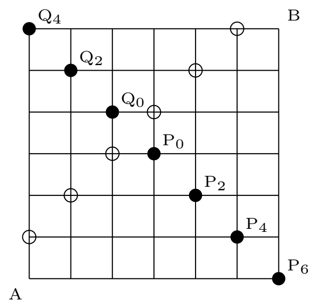
(i) $|\mathcal{L}_\pi| = |\mathcal{R}_\pi| = \binom{n}{k} - \binom{n}{k-1}$,

(ii) $|\mathcal{H}_\pi| = 1$,

(iii) $|\mathcal{D}_\pi| = \left(\binom{n}{k} - \binom{n}{k-1}\right)^2$.

For the proof of this corollary we use the representation of elements of $\mathfrak{TL}_n$ as paths in a square with side n. First we need a small lemma on this path representation:

Figure 5: Sketch for the proof of Lemma 3.9 and Corollary 3.8



Lemma 3.9. *Let $\pi \in \mathfrak{T}\mathfrak{L}_n$ and let $\mathfrak{p}$ be the corresponding shortest path from the bottom left corner (point A) to the upper right corner (point B) below the diagonal AB in a square with side n. Let $P_n, P_{n-2}, \ldots, P_0$ for even n and $P_n, P_{n-2}, \ldots, P_1$ for odd n be the points on the diagonal perpendicular to AB, with P_n in the bottom right corner. Then $\mathfrak{p}$ meets exactly one of the points P_i, and $\mathfrak{p}$ meets the point P_j if and only if $l(\pi) = j$.*

Proof. Figure 5 shows an example of the situation in this proof and the following one. It is obvious that a shortest path from A to B goes above the diagonal if and only if it meets at least one of the circled points in the figure. Hence $\mathfrak{p}$ does not meet any of these points. But then it has to meet any of the points P_i. Since $\mathfrak{p}$ is a shortest path and these points lies on the diagonal perpendicular to AB, $\mathfrak{p}$ meets exactly one of the points P_i.

$l(\pi)$ is the number of sets in π containing both initial and terminal elements. Expressed in terms of parentheses this is the number of brackets opened in the first half and closed in the second half of the expression in $\mathcal{W}_n$. Thus $l(\pi)$ equals the difference of the number of left and right parentheses in the first half of the expression. In terms of paths, the first half of an expression in $\mathcal{W}_n$ corresponds to the path between points A (starting point) and P_i for some i. But the index i is exactly the x-coordinate minus the y-coordinate for the point P_i, that is, by the description in terms of parentheses, $i = l(\pi)$. □

We continue now with the proof of the corollary in the same style as above. Figure 5 gives a picture of the situation also in this proof:

Proof of Corollary 3.8. In the case of (i), by the lemma, the path of any element of $\mathcal{R}_\pi$ with $l(\pi) = i$ meets the point P_i, and since they all are left neighbours the path from A to P_i is fixed. Thus the number of elements in the equivalence class equals the number of paths from P_i to B which does not meet any circled point. The total number of paths from P_i to B equals $\binom{n}{k}$, where $k = \frac{n-l(\pi)}{2}$. Let Q_i be the point P_i reflected in the line of circled points. Then the number of paths from P_i to B that do meet any circled point equals the number of paths from Q_i to B, which can be seen by reflecting the paths in the line of circled points. The number of such paths is $\binom{n}{k-1}$, and hence the result follows.

The second statement is immediate since everything is fixed. In (iii) we are in the same situation as in (i), but the path between A and P_i is not fixed. By symmetry the number of such paths equals the number of paths from P_i to B, and hence the result follows. □

We now turn to the study of Green's relations on $\mathfrak{C}_n$ and $\mathfrak{TLC}_n$. We say that two elements $a, b \in \mathfrak{C}_n$ or $a, b \in \mathfrak{TLC}_n$ are left neighbours provided that $\{x_1, x_2, \ldots, x_m\} \in a$ if and only if $\{x_1, x_2, \ldots, x_m\} \in b$ and there is a set $A \in a$ with $\{x_1, x_2, \ldots, x_l\} \subset A$ if and only if there is a set $B \in b$ with $\{x_1, x_2, \ldots, x_l\} \subset B$. Two elements $a, b \in \mathfrak{C}_n$ are called right neighbours provided that $\{x_1^\circ, x_2^\circ, \ldots, x_m^\circ\} \in a$ if and only if $\{x_1^\circ, x_2^\circ, \ldots, x_m^\circ\} \in b$ and there is a set $A \in a$ with $\{x_1^\circ, x_2^\circ, \ldots, x_l^\circ\} \subset A$ if and only if there is a set $B \in b$ with $\{x_1^\circ, x_2^\circ, \ldots, x_l^\circ\} \subset B$.

As above, two elements are called neighbours if they are both left and right neighbours. Furthermore we let $l(a)$ denote the number of sets in the partition consisting of both initial and terminal elements. Again we call such a set a line, we call a set consisting of only initial elements an initial bracket and a set consisting of only terminal elements a terminal bracket. In this case, $l(a)$ can be viewed as the smallest possible number of "pencil lines" which can be drawn between the left and right hand side to complete the chip. Now we have the following analogue statement of Theorem 3.5:

Theorem 3.10. *Let $a, b \in \mathfrak{C}_n$ or $a, b \in \mathfrak{TLC}_n$. Then*

(i) $a\mathcal{L}b$ if and only if a and b are right neighbours,

(ii) $a\mathcal{R}b$ if and only if a and b are left neighbours,

(iii) $a\mathcal{H}b$ if and only if a and b are neighbours,

(iv) $a\mathcal{D}b$ if and only if $l(a) = l(b)$.

Proof. As in the proof for $\mathfrak{B}_n$ and $\mathfrak{TL}_n$ the necessity follows immediately and the sufficiency of the four statements can be reduced to the sufficiency of the second statement.

For sufficiency of (ii) we prove that $a \in b\mathfrak{C}_n$ for any left neighbours $a, b \in \mathfrak{C}_n$ by construction of an element $c \in \mathfrak{C}_n$ such that $a = bc$. For any set $\{x_1^\circ, x_2^\circ, \ldots, x_m^\circ\} \in b$ we have a set $\{x_1, x_2, \ldots, x_m\} \in c$, and for any set $\{x_1^\circ, x_2^\circ, \ldots, x_m^\circ\} \in a$ we have a set $\{x_1^\circ, x_2^\circ, \ldots, x_m^\circ\} \in c$. Furthermore, if $\{x_1, x_2, \ldots, x_m, y_1^\circ, y_2^\circ, \ldots, y_l^\circ\} \in b$ and $\{x_1, x_2, \ldots, x_m, z_1^\circ, z_2^\circ, \ldots, z_k^\circ\} \in a$, then $\{y_1, y_2, \ldots, y_l, z_1^\circ, z_2^\circ, \ldots, z_k^\circ\} \in a$, and if $a, b \in \mathfrak{TLC}_n$ then $c \in \mathfrak{TLC}_n$. This element c satisfies the above equality, and hence the result follows. □

Again we now have the information needed for calculating cardinalities of the equivalence classes of Green's relations on the semigroups in the theorem. This is stated in the following corollaries:

Corollary 3.11. *Let* $\pi \in \mathfrak{C}_n$ *and let* $l = \max\{1, l(\pi)\}$. *Then*

(i) $|\mathcal{L}_\pi| = |\mathcal{R}_\pi| = l(\pi)! \sum_{k=l}^{n} S_{n,k} \binom{k}{l(\pi)}$,

(ii) $|\mathcal{H}_\pi| = l(\pi)!$,

(iii) $|\mathcal{D}_\pi| = l(\pi)! \left(\sum_{k=l}^{n} S_{n,k} \binom{k}{l(\pi)} \right)^2$.

Proof. As for the previous analogue proof, the involution σ swaps the relations $\mathcal{L}$ and $\mathcal{R}$, and hence the statement for $\mathcal{R}$ follows from that of $\mathcal{L}$.

To construct an element in the class $\mathcal{L}_\pi$ we partition the set of initial elements into at least as many sets as the number of lines $l(a)$, and then choose which of these partitions we will connect to the right side of the chip. If the partition should consist of k sets this can be done in $S_{n,k}\binom{k}{l(\pi)}$ ways. Then we sum over the number of sets in the partition. Finally we draw the lines between initial and terminal elements, which can be done in $l(\pi)!$ ways.

In the construction of a chip in the class $\mathcal{H}_\pi$ the only choice we have to do is to decide how we draw the $l(\pi)$ lines between initial and terminal elements, which can be done in $l(\pi)!$ ways.

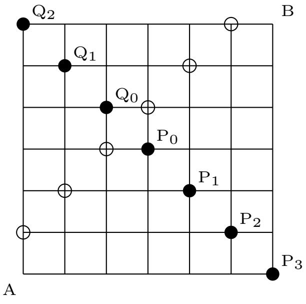
In the case of $\mathcal{D}_\pi$ we have to construct a left hand side and a right hand side of the chip, each of them as the first part of (i). This can be done in $\left(\sum_{k=l}^{n} S_{n,k} \binom{k}{l(\pi)} \right)^2$ ways. Finally we have to draw the connecting lines between the initial and terminal sides of the chip which can be done in $l(\pi)!$ ways, and the statement follows. □

Corollary 3.12. *Let* $\pi \in \mathfrak{TLC}_n$ *with* $l(\pi) = n$. *Then*

$$|\mathcal{L}_\pi| = |\mathcal{R}_\pi| = |\mathcal{H}_\pi| = |\mathcal{D}_\pi| = 1.$$

Proof. Again immediate since there is only one such π with $l(\pi) = n$, namely the identity element. □

Figure 6: Sketch for the proof of Corollary 3.13



Corollary 3.13. *Let* $\pi \in \mathfrak{TLC}_n$ *with* $0 \leq l(\pi) < n$ *and let* $k = n - l(\pi)$. *Then*

(i) $|\mathcal{L}_\pi| = |\mathcal{R}_\pi| = \binom{2n}{k} - \binom{2n}{k-1}$,

(ii) $|\mathcal{H}_\pi| = 1$,

(iii) $|\mathcal{D}_\pi| = \left(\binom{2n}{k} - \binom{2n}{k-1}\right)^2$.

Proof. We will again use the path representation of the elements of $\mathfrak{TLC}_n$ in an analogous construction as the one in Corollary 3.8 and Lemma 3.9. The main difference between this proof and the situation in the analogous proofs above is that an element in $\mathfrak{TLC}_n$ is represented by an expression in $\mathcal{W}_{2n}$ while an element in $\mathfrak{TL}_n$ is represented by an expression in $\mathcal{W}_n$. The difference in number of left and right parentheses in the first half of the expression must the be divided by two, and therefore we get points P_i for every index i, not only for even or odd indices. This behavior totally natural, since the number of lines $l(\pi)$ of an element $\pi \in \mathfrak{TLC}_n$ can be any natural number $0 \leq i \leq n$.

So, in analogy with the above proofs, any shortest path from A to B meet only one of the points P_i, $0 \leq i \leq n$. If the path for the element π meets the point P_i then $l(\pi) = i$. In the case of (i), again the number of elements in the equivalence class equals the number of paths from P_i to B which does not meet any circled point, $l(\pi) = i$. The total number of paths from P_i to B equals $\binom{2n}{k}$, where $k = n - l(\pi)$. The number of paths that do meet any circled point equals the number of paths from Q_i to B, which is $\binom{2n}{k-1}$, and hence the result follows.

In the case $\mathcal{H}_\pi$ everything is fixed, and hence $\mathcal{H}_\pi = \{\pi\}$. In (iii) we are again in the same situation as in (i), but the path between A and P_i is not fixed. As above the number of such paths equals the number of paths from P_i to B, and hence the result follows. $\square$

4 Idempotents and maximal subgroups

The structure of idempotents is another important part in the description of a semigroup. We will see that we have a general connection between idempotents and Green's relations, and between idempotents and maximal subgroups in a semigroup. As in the previous section, we first give some general statements on idempotents and maximal subgroups. Then we consider our four special semigroups and give some combinatorial statements on them. Finally we describe the maximal subgroups of the four semigroups.

Lemma 4.1. *Let e be an idempotent of S. Then e is a right identity element of $\mathcal{L}_e$, a left identity element of $\mathcal{R}_e$ and a two-sided identity element of $\mathcal{H}_e$.*

Proof. Let $a \in \mathcal{L}_e$, then $a = se$ for some $s \in S^1$, and hence $ae = (se)e = se^2 = se = a$. Similarly $a \in \mathcal{R}_e$ implies $a = et$ for some $t \in S$ and $ea = e(et) = e^2t = et = a$. Thus, if $a \in \mathcal{H}_e$, then $ea = a = ae$. □

Lemma 4.2. *Any $\mathcal{H}$-class contains at most one idempotent.*

Proof. Suppose $H = \mathcal{H}_e = \mathcal{H}_f$ for two idempotents e and f. Then, since $e \in \mathcal{H}_f$ we get from the preceding lemma that $fe = e = ef$. Similarly, since $f \in \mathcal{H}_e$ we have $ef = f = fe$, and hence $e = f$. □

Proposition 4.3. *Let $a, b \in S$ be arbitrary. Then $ab \in \mathcal{L}_a \cap \mathcal{R}_b$ if and only if $\mathcal{L}_b \cap \mathcal{R}_a$ contains an idempotent.*

Proof. First suppose $ab \in \mathcal{R}_a \cap \mathcal{L}_b$. Then, since $a \in \mathcal{R}_{ab}$, we may apply Corollary 3.3 to obtain the map $\rho_b\colon x \mapsto xb$, which is a $\mathcal{R}$-class preserving bijection from $\mathcal{L}_a$ onto $\mathcal{L}_{ab} = \mathcal{L}_b$ (since $ab \in \mathcal{L}_b$), mapping a to ab. Since ρ_b is a bijection, there exists $c \in \mathcal{L}_a \cap \mathcal{R}_b$ such that $\rho_b(c) = cb = b$. Furthermore, since $c \in \mathcal{R}_b$, $c = bu$ for some $u \in S^1$. Now $b = cb = bub$ and $c^2 = bubu = bu = c$, and hence c is an idempotent.

Conversely, let $e \in \mathcal{L}_a \cap \mathcal{R}_b$ be an idempotent. Since $e \in \mathcal{R}_b$, $b = es$ for some $s \in S^1$ and we have $eb = e(es) = e^2b = es = b$. Since $e \in \mathcal{L}_a$ we have $ae = a$. Since $e \in \mathcal{R}_b$ and $\mathcal{R}$ is a left congruence, $a = ae\mathcal{R}ab$ and since $e \in \mathcal{L}_a$ and $\mathcal{L}$ is a right congruence, $b = eb\mathcal{L}ab$. Hence $ab \in \mathcal{R}_a \cap \mathcal{L}_b$. □

Corollary 4.4. *Let H be an $\mathcal{H}$-class of a semigroup S. Then the following conditions are equivalent:*

(i) H contains an idempotent.

(ii) There exists $a, b \in H$ such that $ab \in H$.

(iii) H is a maximal subgroup of the semigroup S.

Proof. The implication $(i) \Rightarrow (ii)$ is immediate by taking $a = b = e = e^2$. The converse implication $(ii) \Rightarrow (i)$ follows from the preceding proposition since $H = \mathcal{R}_a \cap \mathcal{L}_b = \mathcal{R}_b \cap \mathcal{L}_a$. The implication $(iii) \Rightarrow (i)$ is immediate since H is a subgroup and the identity in a group is an idempotent.

For the converse implication $(i) \Rightarrow (iii)$, let $e \in H$ be the idempotent in H. Then we have $ea = a = ae$ for all $a \in H$, since $a = se$ for some $s \in S^1$ because $a\mathcal{L}e$, so $ae = see = se = a$, and similarly we have $ae = a$ because $a\mathcal{R}e$. Hence e is the identity element in the subgroup structure of H.

Furthermore, we have to show that H is closed under multiplication. Let $a, b \in H$, that is $a\mathcal{H}e$ and $b\mathcal{H}e$, which implies that $a\mathcal{L}e$ and $b\mathcal{R}e$. Now, $a\mathcal{L}e$ implies $ab\mathcal{L}eb = b$ since $\mathcal{L}$ is a right congruence. Similarly, $b\mathcal{R}e$ implies $ab\mathcal{R}ae = a$ since $\mathcal{R}$ is a left congruence. Hence $ab \in \mathcal{L}_b \cap \mathcal{R}_a = H$.

To show that every element has an inverse we take $a \in H$. Then for all $b \in H$ we have $ab = c \in H$ as above. Then $b\mathcal{L}c$, which implies existence of $s \in S^1$ such that $sb = c$, and since $ab = c$ we may assume $s = a$. By Green's lemma, the map $\sigma \colon x \mapsto sx = ax$ is a $\mathcal{L}$-class preserving bijection $\mathcal{R}_c \to \mathcal{R}_b$. But since it is $\mathcal{L}$-class preserving, it is also a bijection $H \to H$. But then $e \in H$ is in the image of σ, and hence $ax = e$ for some $x \in H$.

Finally we have to show that H is a maximal subgroup. But this follows immediately, since for any two elements g, h in a group G we have $g = (gh^{-1})h$, $h = (hg^{-1})g$, $g = h(h^{-1}g)$ and $h = g(g^{-1}h)$, so $g\mathcal{H}h$. Hence any two elements in a group is contained in the same $\mathcal{H}$-class, and hence the subgroup H is maximal. □

Let us now study the idempotents of $\mathfrak{B}_n$. Here we find notation and some of the statements in [5]. Let $A \subset N = \{1, 2, \dots, n\}$ with $|A| = k$. Let $<$ be a linear order on A and write $A = \{a_1, a_2, \dots, a_k\}$ with $a_{i-1} < a_i$ for $1 < i \leq k$. For an odd $k > 1$ define $e_{A,<}$ to be the element in $\mathfrak{B}_n$ containing the sets $\{a_i, a_{i+1}\}$ for odd i, $1 \leq i < k$, the sets $\{a_i^\circ, a_{i+1}^\circ\}$ for even i, $1 < i < k$, the set $\{a_k, a_1^\circ\}$ and the sets $\{x, x^\circ\}$ for $x \in N \setminus A$.

In a similar way define for even k $e_{A,<}$ to be the element in $\mathfrak{B}_n$ containing the sets $\{a_i, a_{i+1}\}$ for odd i, $1 \leq i < k$, the sets $\{a_i^\circ, a_{i+1}^\circ\}$ for even i, $1 < i < k$, the set $\{a_1^\circ\}, a_k^\circ$ and the sets $\{x, x^\circ\}$ for $x \in N \setminus A$. The elements $e_{A,<}$ are by construction idempotents and are called the elementary idempotents. Furthermore, if $|A| = 2$, the constructed idempotent is called an atom.

Let $f \in \mathfrak{B}_n$ be an idempotent. We make a partition of the set N in to pairwise disjoint subsets, which we will call orbits, in the following way: We say that two elements $x, y \in N$ is in the same orbit if at least one of the sets $\{x, y\}$, $\{x^\circ, y\}$, $\{x, y^\circ\}$ or $\{x^\circ, y^\circ\}$ is contained in f. Using this partition we can prove the following statement about the structure of idempotents in $\mathfrak{B}_n$:

Proposition 4.5. *Every idempotent $f \in \mathfrak{B}_n$ $f \neq e$ can be written as a product*

$$f = e_{A_1,<_1} \cdot e_{A_2,<_2} \cdot \ldots \cdot e_{A_l,<_l}$$

of elementary idempotents where the sets A_i are pairwise disjoint.

Proof. We study each of the orbits of f separately. For each orbit A we construct an element $f_A \in \mathfrak{B}_n$ by copying the part of f corresponding to the orbit $A \in N$. Formally, $\{x, y\} \in f_A$ if and only if $\{x, y\} \in f$, $\{x^\circ, y^\circ\} \in f_A$ if and only if $\{x^\circ, y^\circ\} \in f$, $\{x, y^\circ\} \in f_A$ if and only if $\{x, y^\circ\} \in f$, and finally f_A contains $\{x, x^\circ\}$ for all $x \in N \setminus A$. Since f is an idempotent each orbit chip is an idempotent and hence also f_A. Since f_A acts as the identity element outside the orbit A the product of two such elements for different orbits commute and we have

$$f = \prod_{A \text{ orbit}} f_A.$$

□

Now we turn to the study of some combinatorial properties of the idempotents in $\mathfrak{B}_n$, first some statements on the number of elementary idempotents.

Lemma 4.6. *Let $|A|$ be odd. Then $e_{A,<_A} = e_{B,<_B}$ if and only if $A = B$ and $<_A = <_B$.*

Proof. If both the set and the order is the same, then obviously the idempotents are the same. Conversely, the position of the set consisting of both initial and terminal elements always changes under a cyclic permutation of the order. For other changes of the order, observe that if we swap the order of two terminal pins (which does not change the terminal side of the chip), then we change the initial side of the chip. Hence the result follows. □

Corollary 4.7. *Fix a subset $A \subset M_n$ with $|A| = k$, k odd. Then the total number of pairwise different elementary idempotents on A equals $k!$.*

Proof. Since, by the Lemma, any two different orders on A gives two different idempotents, the number of elementary idempotents on A equals the number of linear orders $<$ on A, that is $n!$. □

For the case of even cardinality we need to define what we mean by a primitive cyclic permutation. Let $L = \{l_1, l_2, \ldots, l_m\}$ be a set linearly ordered by $<$ such that $l_i < l_{i+1}$ for $0 \leq i < m$. Then the map π defined by $l_m \mapsto l_1$ and $l_i \mapsto l_{i+1}$ for $0 \leq i < m$ is called the primitive cyclic permutation of L. Furthermore, the opposite order $<_A^\circ$ of $<_A$ is defined to be the order for which $l_i > l_{i+1}$ for $0 \leq i < m$.

Lemma 4.8. *Let $|A|$ be even. Then $e_{A,<_A} = e_{B,<_B}$ if and only if $A = B$ and one of the following holds:*

(i) $<_B$ is obtained from $<_A$ by an even power of the primitive cyclic permutation.

(ii) $<_B$ is obtained from $<_A^\circ$ by an even power of the primitive cyclic permutation.

Proof. Both sides of the chip consists of cycles of pins connected to the next pin. Also switching to the opposite order gives the same chip since it is symmetric. Therefore it is obvious that if the set is the same and the order $<_B$ differs by an even permutation of the primitive cyclic permutation of either $<_A$ or $<_A^\circ$, then $e_{A,<_A} = e_{B,<_B}$.

Conversely, as in the proof for odd cardinality of the set, any swap of two elements in the order which does not change the terminal side, does change the initial side of the chip, and vice versa. Hence the result follows. □

Corollary 4.9. *Fix a subset $A \subset M_n$ with $|A| = k$, k even. Then the total number of pairwise different elementary idempotents on A equals $(k-1)!$.*

Proof. The total number of linear orders on A is $k!$. Furthermore, for each order $<_A$ there are $\frac{k}{2}$ orders $<_B$ obtained from $<_A$ by an even power of the primitive cyclic permutation, and there are $\frac{k}{2}$ orders $<_B$ obtained from $<_A^\circ$ by an even power of the primitive cyclic permutation. Hence the number of pairwise different elementary idempotents equals $\frac{k!}{k} = (k-1)!$. □

Proposition 4.10. *Let $E(\mathfrak{B}_n)$ be the set of idempotents in $\mathfrak{B}_n$. Then*

$$|E(\mathfrak{B}_n)| = \sum_{\Lambda \vDash M_n} \prod_{\Lambda_i \in \Lambda} f(|\Lambda_i|),$$

where the sum is taken over all partitions $\Lambda = \{\Lambda_1, \dots, \Lambda_k\}$, $\Lambda_i \subset M_n$, $\cup_i \Lambda_i = M_n$ and $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$, and where

$$f(m) = \begin{cases} m! & \text{if } m \text{ is odd,} \\ (m-1)! & \text{if } m \text{ is even.} \end{cases}$$

Proof. We construct an element in $E(\mathfrak{B}_n)$ in the following way: First choose a partition of the set M_n in pairwise disjoint subsets. These subsets will become the orbits of the idempotent. For each such partition we construct the orbits. Each orbit Λ_i can, by Corollary 4.7 and 4.9, be constructed in $|\Lambda_i|!$ respectively $(|\Lambda_i| - 1)!$ different ways for odd respectively even $|\Lambda_i|$. To get the total number of idempotents for this partition we take the product of these numbers over all sets (orbits) in the partition. Hence the total number of idempotents is the sum of these products over all possible partitions. □

This proposition gives a formula to calculate the number of idempotents in $\mathfrak{B}_n$, but since we have to go through all possible partitions it is not very useful for large n. Moreover, this formula does not generalize immediately to the case of $\mathfrak{TL}_n$, since we cannot permute the elements of an orbit with odd number of elements arbitrarily among all elements, without getting a chip which is not contained in $\mathfrak{TL}_n$.

We will continue the combinatorial part of this section with some statements on special types of idempotents, namely those which contains one line or no line at all. For this we let S be any of the semigroups $\mathfrak{B}_n$, $\mathfrak{TL}_n$, $\mathfrak{C}_n$ and $\mathfrak{TLC}_n$, and define $E_k(S)$ to be the set of idempotents $e_i \in S$ such that $l(e_i) = k$.

Proposition 4.11. *The set of idempotents of $\mathfrak{B}_n$ having no lines has cardinality*

$$|E_0(\mathfrak{B}_n)| = \begin{cases} ((n-1)!!)^2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. First we obviously have $l(\pi) \equiv n \mod 2$ for all $\pi \in \mathfrak{B}_n$. Hence $|E_0(\mathfrak{B}_n)| = 0$ for odd n. Furthermore, every element $\pi \in \mathfrak{B}_n$ for which $l(\pi) = 0$ is idempotent, which follows directly from the definition of the multiplication in $\mathfrak{B}_n$. Now, by Theorem 3.5, all elements with the same number of lines is contained in the same $\mathcal{D}$-class, and hence the result is given by inserting $l(\pi) = 0$ in the third formula of Corollary 3.6. □

Proposition 4.12. *The set of idempotents of $\mathfrak{B}_n$ having one line has cardinality*

$$|E_1(\mathfrak{B}_n)| = \begin{cases} (n(n-2)!!)^2 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Proof. As above we have $l(\pi) \equiv n \mod 2$ for all $\pi \in \mathfrak{B}_n$, so $|E_1(\mathfrak{B}_n)| = 0$ for even n. Suppose n is odd. First we claim that every element $\pi \in \mathfrak{B}_n$ with $l(\pi) = 1$ is idempotent. For let $\{i, j^\circ\} \in \pi$ be the line in π. If $i = j$ then we immediately can see that π is idempotent. This follows since we have one orbit $\{i\} \subset N$ of odd cardinality and the other sets consists of either initial or terminal elements. If $i \neq j$, then there exists a $\pi - \pi$ l-connected sequence $x_1, x_2, \ldots, x_{2k+1}$ such that $x_1 = j$ and $x_{2k+1} = i$, hence $\{i, j^\circ\} \in \pi^2$.

Again, all elements with one line is contained in the same $\mathcal{D}$-class and no other elements is contained in the same class, by Theorem 3.5. Inserting $l(\pi) = 1$ into the third formula of Corollary 3.6 gives the desired formula. □

Proposition 4.13. *The set of idempotents of $\mathfrak{TL}_n$ having no lines has cardinality*

$$|E_0(\mathfrak{TL}_n)| = \begin{cases} C_{n/2}^2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Again we have $l(\pi) \equiv n \mod 2$ for all $\pi \in \mathfrak{T}\mathfrak{L}_n$. Hence $|E_0(\mathfrak{T}\mathfrak{L}_n)| = 0$ for odd n. Furthermore, every element $\pi \in \mathfrak{T}\mathfrak{L}_n \subset \mathfrak{B}_n$ for which $l(\pi) = 0$ is idempotent, by the proof above of the analogous statement. Again, all elements with no lines is a $\mathcal{D}$-class, and by Corollary 3.8 with $l(\pi) = 0$ we get the desired result. □

Proposition 4.14. *The set of idempotents of* $\mathfrak{T}\mathfrak{L}_n$ *having one line has cardinality*

$$|E_1(\mathfrak{T}\mathfrak{L}_n)| = \begin{cases} \left(\sum_{i=0}^{k} C_i C_{k-i}\right)^2 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

where $k = \frac{n-1}{2}$.

Proof. For even n there are no elements $\pi \in \mathfrak{T}\mathfrak{L}_n$ with $l(\pi) = 1$, and hence the result follows in this case. For odd n let $\pi \in \mathfrak{T}\mathfrak{L}_n \subset \mathfrak{B}_n$ with $l(\pi) = 0$. Then by the proof of the analogue statement above, π is idempotent. As above we now use Theorem 3.5 and Corollary 3.8 with $l(\pi) = 1$ to get the result. □

In the last two proofs we have used Corollary 3.8 to get the results stated. But in both cases we may calculate the number of elements directly. One can easily see that the left and right hand side of a chip with no lines can be constructed in $|\mathfrak{T}\mathfrak{L}_{n/2}| = C_{n/2}$ different ways respectively. Hence $|E_0(\mathfrak{T}\mathfrak{L}_n)| = C_{n/2}^2$ for even n.

For the case $l(\pi) = 1$ let $\{a, b^\circ\} \in \pi$ be the line. Then, to avoid crossings, both a and b has to be odd. For each choice of a we have to, in order to complete the left hand side of the chip, choose brackets above and below the line. That is, we have to fill gaps of size $a - 1$ and $n - a$ with brackets. This can be done in $|\mathfrak{T}\mathfrak{L}_{a-1}| = C_{(a-1)/2}$ and $|\mathfrak{T}\mathfrak{L}_{n-a}| = C_{(n-a)/2}$ different ways.

Since a is odd, the index for the Catalan numbers varies over the natural numbers smaller or equal $\frac{n-1}{2}$, and hence the left hand side can be constructed in $\sum_{i=0}^{k} C_i C_{k-i}$ number of different ways, where $k = \frac{n-1}{2}$. Obviously, the right hand side is constructed in a totally analogous way, and hence the result in the last proposition follows.

Even if the property that every element with a specified number of lines are idempotent does not generalize, we may generalize this calculation to get the following equality.

Proposition 4.15. *Let* $0 < k \leq \frac{n}{2}$ *and* $l = n - 2k$. *Then*

$$\binom{n}{k} - \binom{n}{k-1} = \sum_{i_1 + \ldots + i_{l+1} = k} \prod_{j=1}^{l+1} C_{i_j}.$$

Proof. We want to construct the left, initial side of an element $\pi \in \mathfrak{TL}_n$ such that $l(\pi) = l$. We assume that $l(\pi) \equiv n \mod 2$, since otherwise we have nothing to construct. By the first part of Corollary 3.8, this can be done in $\binom{n}{k} - \binom{n}{k-1}$ different ways, where $k = \frac{n-l}{2}$. On the other hand, we may calculate this number directly, following the same method as above.

For each possible choice of l pins for the lines we have $n-l$ pins to connect with brackets. These pins will be grouped by the lines into $l+1$ groups with an even number of pins in each group, and where some groups may contain no pins. Let $2i_j$ be the number of pins in the j:th such group. Hence, the number of ways to put brackets in this group is C_{i_j}, so the total number of initial sides of a chip for a specific choice of the lines equals $\prod_{j=1}^{l+1} C_{i_j}$.

To obtain all possible left, initial sides we sum over all possible choices of lines, i.e. all possible choices of the i_j:s such that $\sum_{j=1}^{l+1} 2i_j = n - l$, that is, $\sum_{j=1}^{l+1} i_j = \frac{n-l}{2} = k$. Hence we get the right hand side of the equality, and the statement is proved. □

We continue with $\mathfrak{C}_n$ and $\mathfrak{TLC}_n$ in the case of no lines:

Proposition 4.16. *The set of idempotents of $\mathfrak{C}_n$ having no lines has cardinality*

$$|E_0(\mathfrak{C}_n)| = \left(\sum_{i=1}^{n} S_{n,i}\right)^2.$$

Proof. Since we have no lines, any such element is idempotent by immediate application of the definition of the multiplication. Hence the result follows after insertion of $l(\pi) = 0$ in the third statement of Corollary 3.11. □

The number of elements can also easily be calculated without use of Corollary 3.11. Each side of the chip is constructed by choosing a partition of the pins into k sets for some $1 \leq k \leq n$. For a fixed k this can be done in $S_{n,k}$ number of ways, and hence the total number of possible left hand sides equals the sum $\sum_{i=1}^{n} S_{n,i}$. The right hand side is constructed in the same way, and hence the result follows.

Proposition 4.17. *The set of idempotents of $\mathfrak{TLC}_n$ having no lines has cardinality*

$$|E_0(\mathfrak{TLC}_n)| = C_n^2.$$

Proof. By the first part of the analogue statement for $\mathfrak{C}_n$, every element $\pi \in \mathfrak{TLC}_n$ for which $l(\pi) = 0$ is idempotent. Hence, by Corollary 3.13 with $l(\pi) = 0$, the number of such elements equals C_n^2. □

Finally, we want to study the structure of the maximal subgroups in our four semigroups, and we see that they are all isomorphic to the symmetric group S_k for some k.

Proposition 4.18. *The maximal subgroup $G(e)$ of $\mathfrak{B}_n$ with unit e is isomorphic to the symmetric group $S_{l(e)}$.*

Proof. Let $e \in \mathfrak{B}_n$ be an idempotent. Then, by Corollary 4.4, the $\mathcal{H}$-class of e is a maximal subgroup of $\mathfrak{B}_n$. This maximal subgroup must have identity e, since any $\mathcal{H}$-class can contain at most one idempotent by Lemma 4.2. Any two elements in an $\mathcal{H}$-class is $\mathcal{H}$-related, and hence neighbours. Hence both Ker and Ker$^\circ$ are the same for all elements in the $\mathcal{H}$-class. Now we have a natural bijection f between $\mathcal{H}_e$ and $S_{l(e)}$. Namely, let $\pi \in \mathfrak{B}_n$ and $\mathrm{Im}(\pi) = \{a_1, a_2, \ldots, a_{l(\pi)}\}$ and $\mathrm{Im}^\circ(\pi) = \{b_1^\circ, b_2^\circ, \ldots, b_{l(\pi)}^\circ\}$, both ordered by position in the chip. Then $f(\pi)$ maps the i:th element to the j:th element if and only if there is a line $\{a_i, b_j^\circ\} \in \pi$. Furthermore, f is a bijection, since each orbit contains at most one element of $\mathrm{Im}(\pi)$ and $\mathrm{Im}^\circ(\pi)$ respectively. Hence $G(e)$ is isomorphic to $S_{l(e)}$. □

Proposition 4.19. *The maximal subgroup $G(e)$ of $\mathfrak{C}_n$ with unit e is isomorphic to the symmetric group $S_{l(e)}$.*

Proof. Analogous to the above proof. □

Proposition 4.20. *The maximal subgroup $G(e)$ of $\mathfrak{TL}_n$ with unit e is isomorphic to the trivial group consisting of only the identity element.*

Proof. Let $e \in \mathfrak{TL}_n$ be an idempotent. Then, by Corollary 4.4, the $\mathcal{H}$-class of e is a maximal subgroup of $\mathfrak{TL}_n$, and with identity element e by Lemma 4.2. But, by Corollary 3.8, every $\mathcal{H}$-class of $\mathfrak{TL}_n$ consists of one element only. Hence all maximal subgroups of $\mathfrak{TL}_n$ are isomorphic to the trivial group. □

Proposition 4.21. *The maximal subgroup $G(e)$ of $\mathfrak{TLC}_n$ with unit e is isomorphic to the trivial group consisting of only the identity element.*

Proof. With the last reference changed to the analogue Corollary 3.13, the result follows by the same arguments as above. □

5 Regular semigroups and inverse elements

Let S be a semigroup. Then we say that an element $a \in S$ is regular if $asa = a$ for some $s \in S$. A semigroup S, or more generally any subset $A \in S$, is called regular if all its elements are regular. Furthermore, for two elements $a, b \in S$, b is said to be an inverse of a if $aba = a$ and $bab = b$. Moreover, a is called inverse if it has a unique inverse. Note that $asa = a$ implies that sa and as are idempotents, since $(sa)(sa) = s(asa) = sa$ and $(as)(as) = (asa)s = as$.

As in the preceding sections, we begin this section with some general statements, following [1] and [3].

Lemma 5.1. *An element a in a semigroup S is regular if and only if it has an inverse.*

Proof. It follows immediately from the definitions that if a has an inverse, then a is regular. For the converse statement we need to show that a regular element a has at least one inverse. In particular we show that sas is an inverse of a. So let $b = sas$. Then $aba = a(sas)a = as(asa) = asa = a$ and $bab = (sas)a(sas) = s(asa)(sas) = sa(sas) = s(asa)s = sas = b$. Hence b is an inverse of a. □

Lemma 5.2. *An element a in a semigroup S is regular if and only if the principal left ideal of S generated by a has an idempotent generator e such that $S^1a = S^1e$.*

Proof. First suppose a is regular, that is $asa = a$ for some $s \in S$. Then $e = sa$, by the note above, is an idempotent such that $ae = a$. But this implies that $S^1e \subset S^1a$. Furthermore $e = sa$ implies $S^1a \subset S^1e$ and hence $S^1e = S^1a$. Conversely we assume that for an idempotent e we have $S^1e = S^1a$. Then $a = se$ and $e = ta$ for some $s, t \in S^1$, and we have $ae = see = se = a$ and $a = ae = ata$, so if $t \neq \mathbf{1}$ we have regularity of a. But if $t = \mathbf{1}$, then $e = \mathbf{1}a = a = a^2$ so $a = aaa$ and thus a is regular. □

Lemma 5.3 (Right dual of Lemma 5.2). *An element a in a semigroup S is regular if and only if the principal right ideal of S generated by a has an idempotent generator e such that $aS^1 = eS^1$.*

Proof. Follows immediately from Lemma 5.2 by using the opposite semigroup S^{op} defined in the proof of Lemma 3.3. □

Theorem 5.4. *Let S be a semigroup. Then a $\mathcal{D}$-class D contains a regular element if and only if every element in D is regular. Furthermore, if D is regular, then every $\mathcal{L}$-class and every $\mathcal{R}$-class contained in D contains an idempotent.*

Proof. First observe that Lemma 5.2 in other words states that $a \in S$ is regular if and only if the $\mathcal{L}$-class $\mathcal{L}_a$ contains an idempotent. Hence, if an $\mathcal{L}$-class contains a regular element, then it contains an idempotent, which by the converse implication implies that every element in the $\mathcal{L}$-class is regular. Similarly, Lemma 5.3 states that $a \in S$ is regular if and only if the $\mathcal{R}$-class $\mathcal{R}_a$ contains an idempotent, which implies that if an $\mathcal{R}$-class contains a regular element, then every element in the $\mathcal{R}$-class is regular.

Now suppose $a \in D$ is regular, then every element in $\mathcal{L}_a$ and $\mathcal{R}_a$ is regular. But this implies that every element in D is regular, since every $\mathcal{L}$-class in D meets every $\mathcal{R}$-class in D. Furthermore, if every element in D is regular, then by the reformulations above we immediately have that every $\mathcal{L}$- and $\mathcal{R}$-class of D contains an idempotent. Hence both statements are proved. □

Theorem 5.5. *Let S be a semigroup and $a \in S$ a regular element.*

(i) The set of inverses of a is contained in $\mathcal{D}_a$.

(ii) An $\mathcal{H}$-class $\mathcal{H}_b$ contains an inverse of a if and only if the $\mathcal{H}$-classes $\mathcal{L}_a \cap \mathcal{R}_b$ and $\mathcal{L}_b \cap \mathcal{R}_a$ each contains an idempotent.

(iii) Any $\mathcal{H}$-class contains at most one inverse of a.

Proof. In (i), suppose a' is an inverse of a. Then $a\mathcal{L}a'a$ since $a = s_l a'a$ with $s_l = a$ and $a'a = t_l a$ with $t_l = a'$. Similarly $a'\mathcal{R}a'a$ since $a' = a'as_r$ with $s_r = a'$ and $a'a = a't_r$ with $t_r = a$. Hence $a\mathcal{D}a'$.

For the proof of necessity in statement (ii) we assume that $a' \in \mathcal{H}_b$ is an inverse of a and thus $a'a \in \mathcal{L}_a \cap \mathcal{R}_{a'} = \mathcal{L}_a \cap \mathcal{R}_b$ and $aa' \in \mathcal{L}_{a'} \cap \mathcal{R}_a = \mathcal{L}_b \cap \mathcal{R}_a$ are idempotents by the same arguments as in the proof of (i). For the sufficiency we assume that $e \in \mathcal{L}_a \cap \mathcal{R}_b$ and $f \in \mathcal{L}_b \cap \mathcal{R}_a$ are idempotents. In particular we have $e\mathcal{L}a$ and $f\mathcal{R}a$, which by Lemma 4.1 implies that $ae = a = fa$. Furthermore $e = sa$ and $f = at$ for some $s, t \in S$ (we have $s, t \in S^1$ in general, but since $a = aa'a$ we have $a \in Sa$ and $a \in aS$, and hence $s, t \in S$).

Now define $a' = esf$, which gives $ea' = e^2sf = esf = a'$ and $a'f = esf^2 = esf = a'$. Furthermore $a'a = esfa = esa = e^2 = e$ and then $aa' = aa'f = aa'at = aet = at = f$. From this we get $aa'a = ae = a$ and $a'aa' = a'f = a'$ and hence a and a' are inverses of each other. Furthermore $a'\mathcal{L}f$ since $a' = a'f$ and $aa' = f$, and $a'\mathcal{R}e$ since $a' = ea'$ and $e = a'a$. Hence $a' \in \mathcal{L}_f \cap \mathcal{R}_e = \mathcal{L}_b \cap \mathcal{R}_b = \mathcal{H}_b$.

In (iii), suppose a' and a'' are two inverses of a in an $\mathcal{H}$-class H. Then $a = aa'a$, $a' = a'aa'$, $a = aa''a$ and $a'' = a''aa''$ implies $a\mathcal{L}a'a$, $a'\mathcal{R}a'a$, $a\mathcal{L}a''a$ and $a\mathcal{R}a''a$ respectively. Hence $a'a \in \mathcal{L}_a \cap \mathcal{R}_{a'}$ and $a''a \in \mathcal{L}_a \cap \mathcal{R}_{a''}$. But, since $\mathcal{R}_{a'} = \mathcal{R}_{a''}$ and since $a'a$ and $a''a$ are both idempotents, it follows from

Lemma 4.2 that $a'a = a''a$. Similarly, $aa' \in \mathcal{R}_a \cap \mathcal{L}_{a'}$ and $aa'' \in \mathcal{R}_a \cap \mathcal{L}_{a''}$, and by Lemma 4.2 we conclude that $aa' = aa''$. Hence $a' = a'aa' = a''aa' = a''aa'' = a''$. □

From these general statements we no turn to our four special semigroups, and we find that they contains many regular elements but only a few inverse elements. In fact, all elements are regular:

Proposition 5.6. *The semigroups $\mathfrak{B}_n$, $\mathfrak{TL}_n$, $\mathfrak{C}_n$ and $\mathfrak{TLC}_n$ are all regular.*

Proof. The involution σ defined in the first section, which switches the left and right side of the chip is defined on all these semigroups. It is obvious that for an element a in any of these four semigroups, we have $a = a\sigma(a)a$ and hence a is regular. □

Proposition 5.7. *Any element in $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is inverse.*

Proof. The case $\mathfrak{B}_1$ is trivial since it consists of one element only. In $\mathfrak{B}_2$ there is only one element with no line, hence the $\mathcal{D}$-class of this element is a one element set and the element is inverse. By simple calculation we see that the other two elements are their own inverses. □

Proposition 5.8. *Let $n > 2$. Then an element $a \in \mathfrak{B}_n$ is inverse if and only if $l(a) = n$.*

Proof. First suppose $l(a) = 0$. We know that $\sigma(a)$ is an inverse of a. We now construct another inverse of a in the following way. Let b contain the set $\{i^\circ, j^\circ\}$ if and only if $\{i^\circ, j^\circ\}$ is contained in $\sigma(a)$. Furthermore, choose two initial brackets $\{k, l\}$ and $\{m, n\}$ in $\sigma(a)$. Then let any other initial bracket $\{i, j\}$ be contained in b if and only if $\{i, j\}$ is contained in $\sigma(a)$. Furthermore we let b contain the brackets $\{k, n\}$ and $\{l, m\}$. Then it is obvious that $b \neq \sigma(a)$ is an inverse of a, since the only differences between the products aba and $a\sigma(a)a$ are the dead circles.

Now suppose $0 < l(a) < n$. As above we construct an inverse $b \neq \sigma(a)$ of a, and we construct it from $\sigma(a)$. Let $\{k^\circ, l^\circ\}$ be a terminal bracket and $\{i, j^\circ\}$ be a line in $\sigma(a)$. Then let b contain the line $\{i, k^\circ\}$, the bracket $\{l^\circ, j^\circ\}$ and any other set A if and only if $A \in \sigma(a)$. Then it is immediate that any line with initial element $m \neq i$ is contained in aba if and only if it is contained $a\sigma(a)a$. But since $\{j, i^\circ\} \in a$, $\{i, k^\circ\} \in b$, $\{k, l\} \in a$, $\{l^\circ, j^\circ\} \in b$ and $\{j, i^\circ\} \in a$, we have $\{j, i^\circ\} \in aba$. In the same way $\{i, k^\circ\} \in b$, $\{k, l\} \in a$, $\{l^\circ, j^\circ\} \in b$, $\{j, i^\circ\} \in a$ and $\{i, k^\circ\} \in b$ we have $\{i, k^\circ\} \in bab$. Hence $b \neq \sigma(a)$ is an inverse of a, and the necessity part is proved.

For the sufficiency let $l(a) = n$, then $a \in \mathcal{D}_e \simeq S_n$ by Proposition 4.18, where e is the identity element in $\mathfrak{B}_n$. By Theorem 5.5, any inverse of a

lies in $D_a = D_e \simeq S_n$. Furthermore, if b is an inverse of a in $\mathfrak{B}_n$, then it is also a group inverse in S_n. But then the inverse is unique, and hence a is inverse. □

Proposition 5.9. *Any element in $\mathfrak{TL}_1$ and $\mathfrak{TL}_2$ is inverse.*

Proof. Follows immediately since every element is regular and any $\mathcal{D}$-class consists of one element only. □

Proposition 5.10. *Let $n > 2$. Then an element $a \in \mathfrak{TL}_n$ is inverse if and only if $l(a) = n$.*

Proof. The same arguments as in the proof of the analogous Proposition 5.8 can be used here. For the necessity part, if $l(a) = 0$ we can modify the brackets in such a way that we stay inside $\mathfrak{TL}_n$. In case $0 < l(a) < n$, we choose the line to modify such that its right, terminal endpoint pin lies next to a terminal bracket. Then we certainly does not get any crossings in the chip. Furthermore, the sufficiency is trivial, since the identity is the only element for which $l(a) = n$. □

Proposition 5.11. *Any element in $\mathfrak{C}_1 = \mathfrak{TLC}_1$ is inverse.*

Proof. Follows immediately since every element is regular and any $\mathcal{D}$-class consists of one element only. □

Proposition 5.12. *Let $n > 1$. Then an element $a \in \mathfrak{C}_n$ is inverse if and only if $l(a) = n$.*

Proof. For the necessity, first assume $l(a) = 0$. We know that $\sigma(a)$ is an inverse of a and we want to construct an inverse $b \neq \sigma(a)$ to prove that a is not inverse. Let each initial bracket I be in b if and only if $I \in \sigma(a)$. If the terminal side of $\sigma(a)$ consists of one set T only , then let $\{t\} \in b$ and $T \setminus \{t\} \in b$ for some $t \in T$. Else, the terminal side consists of at least two brackets. Choose two of them, say T_1 and T_2. Then for any T such that $T \neq T_1$ and $T \neq T_2$ we let $T \in b$ if and only if $T \in \sigma(a)$. Finally, let $T_1 \cup T_2 \in b$. It is now easy to see that $b \neq \sigma(a)$ is an inverse of a.

Now suppose $0 < l(a) < n$. We will again construct an inverse $b \neq \sigma(a)$ from $\sigma(a)$. As above, let the structure of the initial side of the chip be the same in b as in $\sigma(a)$. To construct the terminal side, we distinguish between two cases, namely if there exists terminal brackets in $\sigma(a)$ or not. If there exists a terminal bracket in $\sigma(a)$, then in b we connect this bracket to a line in b. If it does not exist any terminal bracket in $\sigma(a)$, then there exists a line containing at least two terminal elements. Choose such a line L and one of the terminal elements $l_t \in L$. Then let b contain $L \setminus \{l_t\}$ and $\{l_t\}$. Certainly, any of these constructions of b gives an inverse of a.

The sufficiency follows by the same arguments as for sufficiency in the analogous Proposition 5.8, since the set, or $\mathcal{D}$-class, of elements with $l(a) = n$ in $\mathfrak{C}_n$ equals that of $\mathfrak{B}_n$. □

Proposition 5.13. *Let $n > 1$. Then an element $a \in \mathfrak{TLC}_n$ is inverse if and only if $l(a) = n$.*

Proof. First note that the sufficiency is trivial, since there is only one element $a \in \mathfrak{TLC}_n$ such that $l(a) = 0$, namely the identity element e. Furthermore, all constructions done in the necessity part of the previous proof can also be used here by choosing brackets and lines next to each other to avoid crossings. □

Acknowledgements

I would like to thank my supervisor Volodymyr Mazorchuk for the idea and presentation of this very interesting subject. I am also grateful for his help, comments and suggestions during the work with this thesis.

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